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On C^*-algebra of a semigroup of partial isometries

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Abstract

A C^*-algebra associated to strongly continuous one-parameter semigroups of partial isometries is introduced as a groupoid C^*-algebra $C^*(\mathcal{G})$. A one-to-one correspondence between non-degenerate representations of $C^*(\mathcal{G})$ and the semigroups is established. It is proved also that $C^*(\mathcal{G})$ contains an ideal J isomorphic to $C_0((0,+\infty],\mathcal{K})$ and $C^*(\mathcal{G})/J \simeq C_0(\mathbb{R})$, where $\mathcal{K}$ is the algebra of compact operators. The C^*-algebra $C^*(\mathcal{G})$ is shown to be generated by one unbounded affiliated element whose image under each non-degenerate representation is the infinitesimal generator of the corresponding semigroup.

Keywords: Semigroups; Partial isometries; Groupoids; Groupoid C^*-algebra; Representations

1. Introduction

Let $t \mapsto U_t$ for $t \geqslant 0$ be a strongly continuous one-parameter semigroup of partial isometries on a Hilbert space H. An example of such a semigroup $t \mapsto R_t^a$ which is not unitary not isometric and not coisometric is given by considering for $a > 0$ the Hilbert space $H = L_2([0,a])$ and

$$\left(R_t^a \xi\right)(x) = \begin{cases} \xi(x-t), & \text{if } t \leqslant x \leqslant a, \\ 0, & \text{if } x < t, \end{cases}$$

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with the understanding that $R_t^a = 0$ if $t > a$. Any semigroup $t \mapsto U_t$ which is unitarily equivalent to R_t^a is called a truncated shift.

Strongly continuous one-parameter semigroups of partial isometries were studied in [8,12], where a complete structure theorem was obtained. It asserts, in analogy with the Wold decomposition of single isometry, that for $t \mapsto U_t$, $t \geqslant 0$, there are Hilbert spaces H_1, H_+, H_-, H_0 and there is a unitary operator S such that

$$SU_tS^* = V_t \oplus K_t \oplus S_t \oplus T_t,$$

where V_t is unitary on H_1, K_t is purely isometric on H_+, S_t is purely coisometric on H_- and T_t on H_0 is a direct integral of truncated shifts. The structure of more general centered one-parameter semigroups where studied in [12].

Recall that there exists a unique irreducible semigroup of isometries $t \mapsto K_t$ up to unitary equivalence which is given by $(K_t\xi)(x) = \chi_{[t,+\infty)}(x)\xi(x-t)$, $\xi \in L_2([0,+\infty))$. The semigroup is important in the study of Wiener–Hopf equations; i.e., equations of the form

$$\bigl(I + W(f)\bigr)\xi = \eta,$$

where $f \in L_1(\mathbb{R})$, $\eta, \xi \in L_2([0,+\infty))$ and where $W(f)$ is the Wiener–Hopf operator defined by the formula

$$\bigl(W(f)\xi\bigr)(x) = \int_{\mathbb{R}} f(t)(K_t\xi)(x)\,dt = \int_0^{+\infty} f(x-t)\xi(t)\,dt.$$

The Wiener–Hopf C^*-algebra $\mathcal{W}$ generated by all Wiener–Hopf operators is the universal C^*-algebra for semigroups of isometries in the sense that every its non-degenerate representation $\pi : \mathcal{W} \to B(H)$ arises from unique semigroup of isometries $t \mapsto U_t$: $\pi(W(f)) = \int_{\mathbb{R}} f(t)U_t\,dt$ [4,5]. This C^*-algebra and some its generalisations were studied intensively in the literature in connection with numerous applications [1,4,5,10]. There is no approach in the literature to associate a C^*-algebra to a general strongly continuous one-parameter semigroup $t \mapsto U_t$, $t \geqslant 0$, of partial isometries (for the case of discrete semigroups $t \mapsto U_t$ see [9]). The aim of this work is to fill up this gap. We will use a groupoid C^*-algebras approach which was introduced by Muhly and Renault [10] in connection with the study of the C^*-algebras of Wiener–Hopf operators on normal subsemigroups of locally-compact groups.

Consider for $t \mapsto U_t$, $t \geqslant 0$, the C^*-algebra $\mathfrak{A}(U_t\colon\, t \geqslant 0)$ generated by Wiener–Hopf type operators $\int_{\mathbb{R}} f(t)U_t\,dt$ ($f \in L_1(\mathbb{R})$, $U_{-t} = U_t^*$, $t > 0$). In this paper we show that for U_t which is a direct integral of R_t^a with respect to the Lebesgue measure on $(0,+\infty)$, $\mathfrak{A}(U_t\colon\, t \geqslant 0)$ is the image of a faithful representation of a groupoid C^*-algebra $C^*(\mathcal{G})$ for a certain locally compact groupoid $\mathcal{G}$. We establish a one-to-one correspondence between the representations of $C^*(\mathcal{G})$ and strongly continuous one-parameter semigroups of partial isometries up to a natural action of $\mathbb{Z}^2$ given by the involution: $t \mapsto U_t$, $t \geqslant 0$, is a semigroup of partial isometries iff so is the family $t \mapsto U_t^*$, $t \geqslant 0$. To obtain this fact we established the exact sequence

$$0 \to C_0\bigl((0,+\infty],\mathcal{K}\bigr) \to C^*(\mathcal{G}) \to C_0(\mathbb{R}) \to 0.$$

Here $\mathcal{K}$ is the algebra of compact operators. Finally we found an unbounded generator A of the C^*-algebra $C^*(\mathcal{G})$ in the sense of [16]. It has the property that for each non-degenerated representation π of $C^*(\mathcal{G})$ the operator $\pi(A)$ is the generator of the corresponding one-parameter semigroup of partial isometries.

2. Preliminary

For $a > 0$ consider the following strongly continuous one-parameter family of partial isometries acting on $L_2([0,a])$:

$$\left(U_t^a \xi\right)(x) = \begin{cases} \chi_{[t,a]}(x)\xi(x-t), & \text{if } t \geqslant 0, \\ \chi_{[0,a+t]}(x)\xi(x-t), & \text{if } t < 0, \end{cases}$$

where χ_A is the indicator of the set A. One can check that $U_t^a U_s^a = U_{t+s}^a$, $(U_t^a)^* = U_{-t}^a$ for all $t, s \in \mathbb{R}^+$ and that U_t^a is a partial isometry with the initial projection $M_{\chi_{[0,a-t]}}$ and the range projection $M_{\chi_{[t,a]}}$ (for $t > 0$). Here M_g is the multiplication operator on $L_2([0,a])$ by $g \in L_\infty([0,a])$. We also denote by $\{U_t^\infty\}_{t\in\mathbb{R}}$ the strongly continuous family of isometries acting on $L_2([0,+\infty))$ by

$$\left(U_t^\infty \xi\right)(x) = \begin{cases} \chi_{[t,+\infty)}(x)\xi(x-t), & \text{if } t \geqslant 0, \\ \xi(x-t), & \text{if } t < 0. \end{cases}$$

By [8,12] any irreducible strongly continuous one-parameter semigroup of partial isometries, $\{u_t\}_{t\geqslant 0}$, $u_0 = 1$, on a Hilbert space is unitarily equivalent either to $\{U_t^a\}_{t\geqslant 0}$, $a \in \mathbb{R}^+ \cup \{\infty\}$ or to $\{(U_t^\infty)^*\}_{t\geqslant 0} = \{U_{-t}^\infty\}_{t\geqslant 0}$ or it is a one-dimensional unitary semigroup $\{e^{it\phi}\}_{t\geqslant 0}$, $\phi \in \mathbb{R}$.

Note that $\{(U_t^a)^*\}_{t\geqslant 0}$, $0 < a < \infty$, is also an irreducible strongly continuous one-parameter semigroup of partial isometries. It is unitarily equivalent to $\{U_t^a\}_{t\geqslant 0}$, the equivalence is given by $(W\xi)(x) = \xi(a-x)$.

To each $f \in L_1(\mathbb{R})$ let us associate an operator $W_a(f)$ on $L_2([0,a])$ if $0 < a < \infty$ or on $L_2([0,+\infty))$ if $a = \infty$ given by

$$\left(W_a(f)\xi\right)(x) = \int\limits_{\mathbb{R}} f(s) U_s^a \xi(x)\, ds = \int\limits_{x-a}^{x} f(s)\xi(x-s)\, ds$$

($x - \infty := -\infty$ for any $x \in \mathbb{R}$). The operators $W_\infty(f)$ are known as Wiener–Hopf operators. The C^*-algebra generated by all those operators, also called the Wiener–Hopf C^*-algebra, was extensively studied in the literature, in particular, because of its numerous applications, see [1,3–5,7]. In [10] this algebra was realized as the image of a groupoid C^*-algebra under a suitable representation. Our aim is to associate a groupoid C^*-algebra to the semigroup of partial isometries following an idea similar to one in [10]. For this let us calculate first the product $W_a(f_1)W_a(f_2)$, $f_1, f_2 \in L_1(\mathbb{R})$ to find the following:

$$\begin{aligned} \left(W_a(f_1)W_a(f_2)\xi\right)(x) &= \int\limits_{x-a}^{x} f_1(t) \int\limits_{x-t-a}^{x-t} f_2(s)\xi(x-t-s)\, ds\, dt \\ &= \int\limits_{x-a}^{x} \int\limits_{x-a}^{x} f_1(t) f_2(s-t)\xi(x-s)\, ds\, dt \end{aligned}$$

$$= \int_{x-a}^{x} \left(\int_{x-a}^{x} f_1(t) f_2(s-t)\,dt \right) \xi(x-s)\,ds$$
$$= \int_{x-a}^{x} h(x,s,a)\xi(x-s)\,ds,$$

where $h(x,s,a) = \int_{x-a}^{x} f_1(t) f_2(s-t)\,dt$. Take now $f_i \in L_1(\mathbb{R})$, $i = 1,2,3,4$, and let h_1 and h_2 correspond to $W_a(f_1)W_a(f_2)$ and $W_a(f_3)W_a(f_4)$, respectively. Then we have

$$\big(W_a(f_1)W_a(f_2)W_a(f_3)W_a(f_4)\xi\big)(x)$$
$$= \int_{x-a}^{x} h_1(x,s,a) \int_{x-s-a}^{x-s} h_2(x-s,u,a)\xi(x-s-u)\,du\,ds$$
$$= \int_{x-a}^{x} h_1(x,s,a) \int_{x-a}^{x} h_2(x-s,u-s)\xi(x-u)\,du\,ds$$
$$= \int_{x-a}^{x} h(x,u,a)\xi(x-u)\,du,$$

where

$$h(x,u,a) = \int_{x-a}^{x} h_1(x,s,a)h_2(x-s,u-s,a)\,ds. \tag{1}$$

Given $a \in (0,\infty]$, let $X_a = [0,a]$ if $a > 0$ and $X_a = [0,+\infty)$ if $a = \infty$. Consider the collection $\mathfrak{H}$ of all functions h on $[0,+\infty) \times \mathbb{R} \times (0,+\infty]$ such that for each $a \in (0,+\infty]$ $\operatorname{supp} h(\cdot,\cdot,a) \subseteq X_a$ and

$$\operatorname{ess\,sup} \left\{ \int_{x-a}^{x} h(x,s,a)\,ds \;\middle|\; a > 0 \cup a = \infty,\ x \in [0,a] \right\} \tag{2}$$

is finite. An easy calculation shows that $W_a(h)^* = W_a(h^*)$, where

$$h^*(x,s,a) = \overline{h(x-s,-s,a)}.$$

With respect to the product defined by (1) and expression (2) as norm $\mathfrak{H}$ becomes a $*$-Banach algebra. The operators

$$\big(W_a(h)\xi\big)(x) = \int_{x-a}^{x} h(x,s,a)\xi(x-s)\,ds, \quad h \in \mathfrak{H},$$

define a representation of $\mathfrak{H}$.

3. Groupoids and their C^*-algebras

Let us recall some basic definitions and facts from the theory of groupoids and groupoid C^*-algebras (for detailed treatment see [13,14]).

A *groupoid* is a set $\mathfrak{G}$ together with a subset $\mathfrak{G}^2 \subseteq \mathfrak{G} \times \mathfrak{G}$, a product map $(a, b) \to ab$ from $\mathfrak{G}^2$ to $\mathfrak{G}$ and an inverse map $a \to a^{-1}$ (so that $(a^{-1})^{-1} = a$) such that:

(i) if $(a, b), (b, c) \in \mathfrak{G}^2$ then $(ab, c), (a, bc) \in \mathfrak{G}^2$ and

$$(ab)c = a(bc).$$

(ii) $(b, b^{-1}) \in \mathfrak{G}^2$ for all $b \in \mathfrak{G}$ and if $(a, b) \in \mathfrak{G}^2$ then

$$a^{-1}(ab) = b, \quad (ab)b^{-1} = a.$$

If, in addition, $\mathfrak{G}$ is a locally compact Hausdorff space and if the maps defined above are continuous ($\mathfrak{G}^2$ is endowed with relative topology) then $\mathfrak{G}$ is called a *locally compact* groupoid. The map $d : \mathfrak{G} \to \mathfrak{G}, x \mapsto x^{-1}x$, is called the *domain* map. The map $r : \mathfrak{G} \to \mathfrak{G}, x \mapsto xx^{-1}$, is called the *range* map. They have a common image $\mathfrak{G}^0$ called the *unit space* of $\mathfrak{G}$.

If $\mathfrak{G}$ is a groupoid and $u \in \mathfrak{G}^0$, then $d^{-1}(u) \cap r^{-1}(u)$ becomes a locally compact group with identity u. This group is called the *isotropy group* at u. Two points u and v in $\mathfrak{G}^0$ lie in the same orbit if and only if $d^{-1}(u) \cap r^{-1}(v) \neq \emptyset$. For $u \in \mathfrak{G}^0$ the orbit $\mathrm{Orb}(u)$ through u is the set of units that lie in the same orbit with u. A set which is the union of orbits is called *invariant*. A groupoid is called *transitive* if there is only one orbit and *principal* if the map $x \mapsto (d(x), r(x))$ is one-to-one. Let $C_c(\mathfrak{G})$ be the space of compactly supported continuous complex-valued functions defined on a groupoid $\mathfrak{G}$. A family of measures $\{\lambda^u\}$ on $\mathfrak{G}$ indexed by $\mathfrak{G}^0$ is called a (left) *Haar system* on $\mathfrak{G}$ if:

(i) each λ^u is supported by $r^{-1}(u)$,
(ii) for each $f \in C_c(\mathfrak{G})$ the function $u \mapsto \int f\, d\lambda^u$ is continuous on $\mathfrak{G}^0$,
(iii) for each $x \in \mathfrak{G}$ and $f \in C_c(\mathfrak{G})$, $\int f(xy)\, d\lambda^{d(x)}(y) = \int f(y)\, d\lambda^{r(x)}(y)$.

Next we want to associate a groupoid with the algebra $\mathfrak{H}$ from Section 2. Let $\overline{\mathbb{R}}$ denote $(-\infty, \infty]$. Consider the space $\mathcal{G} = \{(x, s, a) \in [0, \infty] \times \mathbb{R} \times (0, \infty]:\ x \in [0, a],\ x - s \in [0, a]\}$ with the usual topology on it.

The space becomes a locally compact groupoid if we set $\mathcal{G}^2 = \{[(x, s, a), (y, t, b)] \mid y = x - s, a = b\}$ and define the product and the inverse maps by

$$(x, s, a) \cdot (x - s, t, a) = (x, s + t, a), \quad (x, s, a)^{-1} = (x - s, -s, a).$$

(We use the convention $\infty - s = \infty$ for each $s \in \mathbb{R}$.) The maps d and r satisfy the equations

$$d(x, s, a) = (x, s, a)^{-1}(x, s, a) = (x - s, 0, a), \tag{3}$$

$$r(x, s, a) = (x, s, a)(x, s, a)^{-1} = (x, 0, a). \tag{4}$$

The unit space of $\mathcal{G}$ is $\{(x, 0, a):\ x \in [0, a],\ a > 0\} \cup \{(x, 0, \infty):\ x \geqslant 0\} \cup \{\infty, 0, \infty\}$.

Let $C_c(\mathcal{G})$ be the space of compactly supported, continuous, complex-valued functions defined on $\mathcal{G}$. We may take $\lambda^u = \chi_{\mathcal{G}}\delta_x \times \lambda \times \delta_a$ for $u = (x, 0, a)$, where δ_x is a point mass at x and λ is the Lebesgue measure on $\mathbb{R}$, and obtain a left Haar system on $\mathcal{G}$. A multiplication, involution and a norm with respect to which $C_c(\mathcal{G})$ becomes a normed $*$-algebra are defined as follows: for $u = (x, s, a)$, with $a, x \neq \infty$, $f, g \in C_c(\mathcal{G})$ the product is given by

$$(f * g)(u) = \int f(uv)g\left(v^{-1}\right) d\lambda^{d(u)}(v) = \int_{x-s-a}^{x-s} f(x, s+t, a)g(x-s-t, -t, a)\, dt$$

$$= \int_{x-a}^{x} f(x, r, a)g(x-r, s-r, a)\, dr,$$

otherwise

$$(f * g)(u) = \begin{cases} \int_{-\infty}^{x} f(x, r, \infty)g(x-r, s-r, \infty)\, dr, & u = (x, s, \infty), 0 \leqslant x < \infty, \\ \int_{-\infty}^{+\infty} f(\infty, r, \infty)g(\infty, s-r, \infty)\, dr, & u = (\infty, s, \infty); \end{cases}$$

the involution and the norm are given by the formulas

$$f^*(u) = \overline{f\left(u^{-1}\right)} = \overline{f(x-s, -s, a)}$$

and

$$\|f\|_I = \max\left\{ \sup_{u \in \mathcal{G}^0} \int |f|\, d\lambda^u, \sup_{u \in \mathcal{G}^0} \int |f^*|\, d\lambda^u \right\}$$

$$= \max\left\{ \sup\left\{ \int_{x-a}^{x} \left|f(x, t, a)\right| dt, \int_{x-a}^{x} \left|f(x-t, -t, a)\right| dt\colon\ (x, a) \in [0, a] \times (0, \infty] \right\}\right\}.$$

Let $L^I(\mathcal{G}, \lambda)$ be the completion of $C_c(\mathcal{G})$ with respect to the defined norm. We denote by $C^*(\mathcal{G})$ (or $C^*(\mathcal{G}, \lambda)$) its enveloping C^*-algebra.

4. Structure and representations of the groupoid C^*-algebra $C^*(\mathcal{G})$

The aim of this section is to show that there is a one-to-one correspondence between representations of the defined groupoid C^*-algebra and strongly continuous semigroups of partial isometries up to the action of $\mathbb{Z}_2$ defined by involution: if $\{u_t\}_{t \geqslant 0}$ is a strongly continuous semigroup of partial isometries so is the family $\{u_t^*\}_{t \geqslant 0}$.

In what follows we denote by $C_c(X)$ the algebra of all continuous complex-valued functions on a locally compact set X with compact support and by $C_0(X)$ the continuous complex-valued functions vanishing at infinity.

For $g \in C_c(\mathbb{R} \times (0, +\infty])$ we define $\tilde{g} \in C_c(\mathcal{G})$ by $\tilde{g}(x, s, a) = g(s, a)$.

Proposition 1. *The subalgebra $\mathcal{A}$ generated by $\{\tilde{g}\colon\ g \in C_c(\mathbb{R} \times (0, \infty])\}$ is dense in $C^*(\mathcal{G})$.*

Proof. It is enough to prove that $\mathcal{A}$ is dense in $C_c(\mathcal{G})$ endowed with the inductive limit topology. Let $D = [0, \infty] \times (0, \infty]$. We claim first that the algebra generated by $\psi_f(y, a) = \int_{y-a}^{y} f(t)\,dt$, $f \in C_c(\mathbb{R})$ is dense in $C_0(D)$. The family of functions $\{\psi_f\}_{f \in C_c(\mathbb{R})}$ separates the points of D. Indeed if $(y, a), (x, b) \in D$ and $\int_{y-a}^{y} f(t)\,dt = \int_{x-b}^{x} f(t)\,dt$ for all functions $f \in C_c(\mathbb{R})$ then $[y-a, y] = [x-b, x]$ and therefore $(y, a) = (x, b)$. The family $\{\psi_f\}$ does not vanish at any point of $(y, a) \in D$ since otherwise $\int_{y-a}^{y} f(t)\,dt = 0$ for all $f \in C_c(\mathbb{R})$ which is impossible if $a > 0$. Clearly, the family is closed under complex conjugation. Thus, by the Stone–Weierstrass theorem the algebra generated by $\{\psi_f\}$ is dense in $C_0(D)$. Hence to prove the statement it is sufficient to approximate the functions of the form $\varphi(x, s, a) = (\int_{x-a}^{x} f(t)\,dt)g(s)$, where $f \in C_c(\mathbb{R})$, $g \in C_c(\mathbb{R} \times (0, \infty])$, by the elements of $\mathcal{A}$. Fix now $f \in C_c(\mathbb{R})$, $g \in C_c(\mathbb{R} \times (0, \infty])$ and denote by K a compact neighborhood of $(\operatorname{supp} g)_1$, here $(\operatorname{supp} g)_1 = \{s\colon (s, a) \in \operatorname{supp} g \text{ for some } a \in (0, \infty]\}$. Fix $\varepsilon > 0$ and let $M = \int_{\mathbb{R}} |f(t)|\,dt$ which is finite since f has a compact support. Then there exists a neighborhood $U \subset \mathbb{R}$ of 0 such that $U + \operatorname{supp}(g)_1 \subset K$ and $|g(t, a) - g(s, a)| < \varepsilon/M$ whenever $t - s \in U$. Since $\operatorname{supp}(f)$ is compact, there exists $\{t_i\}_{i=1}^{n}$, such that $\{U + t_i\}_{i=1}^{n}$ is an open covering of $\operatorname{supp}(f)$. Let $\{h_i\}_{i=1}^{n}$ be a partition of unity of $\operatorname{supp}(f)$ such that $\operatorname{supp}(h_i) \subset U + t_i$. We have $\sum_{i=1}^{n} h_i(r) = 1$ for $r \in \operatorname{supp}(f)$. The set $(\operatorname{supp} g)_2 = \{a\colon (s, a) \in \operatorname{supp} g$ for some $s \in \mathbb{R}\}$ is compact. Let θ be a function in $C_c(\mathbb{R})$ such that $\theta(a) = 1$ for $a \in (\operatorname{supp} g)_2$. Set

$$f_i(r, a) = f(r)\theta(a)h_i(r), \qquad g_i(r, a) = g(r + t_i, a)$$

and

$$\psi = \sum_{i=1}^{n} \tilde{f}_i * \tilde{g}_i.$$

We have

$$\begin{aligned}
\varphi(x, s, a) - \psi(x, s, a) &= \left(\int_{x-a}^{x} f(v)\,dv\right) g(s, a) - \left(\sum_{i=1}^{n} \tilde{f}_i * \tilde{g}_i\right)(x, s, a) \\
&= \int_{x-a}^{x} \left[f(v)g(s, a) - \sum_{i=1}^{n} \tilde{f}_i(x, v, a)\tilde{g}_i(x - v, s - v, a) \right] dv \\
&= \int_{x-a}^{x} \left[f(v)g(s, a) - \sum_{i=1}^{n} f_i(v)g_i(s - v, a) \right] dv \\
&= \sum_{i=1}^{n} \int_{x-a}^{x} f_i(v, a)\big(g(s, a) - g_i(s - v, a)\big)\,dv \\
&= \sum_{i=1}^{n} \int_{x-a}^{x} f(v)h_i(v)\big(g(s, a) - g(t_i - v + s, a)\big)\,dv.
\end{aligned}$$

Moreover, $|g(s,a)-g(t_i-v+s,a)|<\varepsilon/M$ as $s-(t_i-v+s)=v-t_i\in U$ for $v\in \operatorname{supp}(f_i)\subset \operatorname{supp}(h_i)\subset t_i+U$. Thus

$$\left|\varphi(x,s,a)-\psi(x,s,a)\right|<(\varepsilon/M)\int_{-a}^{a}\left|f(v)\right|dv<\varepsilon.$$

It is left to see that ψ and φ have common compact support. Clearly, φ is supported in $[0,\infty]\times K\times(\operatorname{supp} g)_2$. Since fh_i is supported in $U+t_i$ then the integrand $f_i(v,a)g_i(s-v,a)=f(v)h_i(v)g(t_i+s-v,a)$ vanishes whenever $v\notin U+t_i$. Further $t_i+s-v\in(\operatorname{supp} g)_1$ if and only if $s\in v-t_i+(\operatorname{supp} g)_1$. Therefore if $v\in U+t_i$ and $t_i+s-v\in(\operatorname{supp} g)_1$ then $s\in U+t_i-t_i+(\operatorname{supp} g)_1\subset U+(\operatorname{supp} g)_1\subset K$. Thus, ψ is also supported in $[0,\infty]\times K\times(\operatorname{supp} g)_2$. This completes the proof. □

For the chosen left Haar system $\{\lambda^u\}$ on $\mathcal{G}$ let $\lambda_u=(\lambda^u)^{-1}$ be the image of λ^u under the map $x\to x^{-1}$. Then every Radon measure μ on $\mathcal{G}^0$ induces two measures ν and ν^{-1} on $\mathcal{G}$ by the rules $\nu=\int\lambda^u\,d\mu(u)$, $\nu^{-1}=\int\lambda_u\,d\mu(u)$.

The Hilbert space $L^2(\nu^{-1})$ carries a representation of $C^*(\mathcal{G})$ which is called the representation *induced off the unit space by* μ and is denoted by $\operatorname{Ind}\mu$. It is defined by the formula

$$\operatorname{Ind}\mu(f)\xi(x)=\int f(xy)\xi\left(y^{-1}\right)d\lambda^{d(x)}(y),$$

where $f\in C_c(\mathcal{G})$, $\xi\in L^2(\nu^{-1})$. Since $\|\operatorname{Ind}\mu(f)\|\leqslant\|f\|_{L^I}$ the representation $\operatorname{Ind}\mu$ extends to $C^*(\mathcal{G})$.

Let $u=(0,0,a)\in\mathcal{G}^0$ and let δ_u be the point mass at $\{u\}$. Denote the induced measures by ν_u and ν_u^{-1}. Then for $f\in C_c(\mathcal{G})$ we have

$$\int f\,d\nu_u=\int_{-a}^{0}f(0,s,a)\,ds\quad\text{and}\quad\int f\,d\nu_u^{-1}=\int_{0}^{a}f(s,s,a)\,ds,$$

giving $\operatorname{supp}\nu_u^{-1}=\{(s,s,a)\colon s\in\mathbb{R}\}$. For $\xi\in L^2(\nu_u^{-1})$ and $f\in C_c(\mathcal{G})$ we have

$$\left(\operatorname{Ind}\delta_u(f)\xi\right)(s,s,a)=\int_{0}^{a}f(s,s-t,a)\xi(t,t,a)\,dt.\tag{5}$$

The operator $U:L^2([0,a])\to L^2(\nu_u^{-1})$ defined by the formula

$$(U\xi)(x,t,b)=\begin{cases}\xi(t), & x=t,\ b=a,\\ 0, & \text{otherwise},\end{cases}$$

where $\xi\in L^2([0,a],\lambda)$ is an isometry with the inverse $(U^{-1}\zeta)(t)=\zeta(t,t,a)$ $(\zeta\in L^2(\nu_u^{-1}))$. For $f\in C_c(\mathcal{G})$ and $\xi\in L^2(\nu_u^{-1})$ we have

$$\begin{aligned}
\big(UW_a(f)U^{-1}\xi\big)(t,t,a) &= \big(W_a(f)U^{-1}\xi\big)(t) \\
&= \int_{t-a}^{t} f(t,s,a)\big(U^{-1}\xi\big)(t-s)\,ds \\
&= \int_{t-a}^{t} f(t,s,a)\xi(t-s,t-s,a)\,ds \\
&= \int_{0}^{a} f(t,t-s,a)\xi(s,s,a)\,ds.
\end{aligned}$$

Comparing with (5) we obtain that $\operatorname{Ind}\delta_u$ is unitarily equivalent to W_a. Moreover, by [10, Proof of Proposition 2.17], if $u, v \in \mathcal{G}_0$ lie in the same orbit then $\operatorname{Ind}\delta_u$ and $\operatorname{Ind}\delta_v$ are unitarily equivalent, so that for any $a \in (0,\infty]$ and $x, y \neq \infty$, $\operatorname{Ind}\delta_{(x,0,a)} \simeq \operatorname{Ind}\delta_{(y,0,a)}$. That W_a, $a \in (0,\infty]$, is irreducible can be easily checked.

If $u = (\infty, 0, \infty)$, then for $f \in C_c(\mathcal{G})$, $\int f\, d\nu_u^{-1} = \int_{\mathbb{R}} f(\infty, s, \infty)\, ds$ and for $\xi \in L_2(\nu_u^{-1})$ we have

$$\big(\operatorname{Ind}\delta_u(f)\xi\big)(\infty,s,\infty) = \int_{\mathbb{R}} f(\infty, s-t, \infty)\xi(\infty,t,\infty)\,dt = (\tilde{f} * \tilde{\xi})(s),$$

where $\tilde{f}(s) = f(\infty, t, \infty)$, and $*$ is the usual convolution.

Consider also the following family of one-dimensional representations ω_ϕ, $\phi \in \mathbb{R}$, of $C^*(\mathcal{G})$: for $f \in C_c(\mathcal{G})$,

$$\omega_\phi(f) = \int_{\mathbb{R}} f(\infty, t, \infty)\, e^{-it\phi}\, dt = \widehat{\tilde{f}}(\phi),$$

where $\widehat{\tilde{f}}$ is the Fourier transform of $\tilde{f}$. Those representations come from one-dimensional strongly continuous one-parameter semigroups of partial isometries = unitaries. We have that $\operatorname{Ind}\delta_{(\infty,0,\infty)}$ is unitarily equivalent to the direct integral $\int_{\mathbb{R}}^{\oplus} \omega_\phi\, d\phi$ of these one-dimensional representations via the Fourier transform on $\mathbb{R}$.

Note also that by [10, Proposition 2.13], $\operatorname{Ind}\mu \simeq \int_{\mathcal{G}^0}^{\oplus} \operatorname{Ind}\delta_u\, d\mu$.

Let $C^*_{\mathrm{red}}(\mathcal{G},\lambda)$ be the reduced groupoid C^*-algebra of $\mathcal{G}$. Recall that it is obtained by the completion of $C_c(\mathcal{G},\lambda)$ with respect to the norm $\|f\|_{\mathrm{red}} = \sup \|L(f)\|$, $f \in C_c(\mathcal{G})$, where L ranges over all representations induced off the unit space [14, Chapter II, Definition 2.8]. By [10, Proposition 2.15], $C^*_{\mathrm{red}}(\mathcal{G},\lambda) \cong C^*(\mathcal{G},\lambda)$.

Proposition 2. *Let m be a probability measure on $\mathbb{R}^+$ which is equivalent to the Lebesgue measure. Then the representation $\int_{a>0}^{\oplus} \operatorname{Ind}\delta_{(0,0,a)}\, dm(a)$ of $C^*(\mathcal{G})$ is faithful.*

Proof. Let λ be our Haar system for $\mathcal{G}$. By [10, Proposition, p. 12], for a measure μ on $\mathcal{G}^0$ the kernel of $\operatorname{Ind}\mu = \int_{\mathcal{G}^0}^{\oplus} \operatorname{Ind}\delta_u\, d\mu(u)$ in $C^*_{\mathrm{red}}(\mathcal{G},\lambda) = C^*(\mathcal{G},\lambda)$ is the ideal $C^*_{\mathrm{red}}(\mathcal{G}|_U, 1_U\lambda)$, where U is the largest open invariant subset of $\mathcal{G}^0$ such that $\mu(U) = 0$. So $\operatorname{Ind}\mu$ has the trivial kernel if for any open invariant subset U of $\mathcal{G}^0$ we have $\mu(U) \neq 0$. As $\mathcal{G}^0 = \{(x,0,a)\colon x \in [0,a],\ a > 0\} \cup \{(x,0,\infty)\colon x \geqslant 0\} \cup \{(\infty,0,\infty)\}$ and $\operatorname{Orb}((x,0,a)) = \{(y,0,a)\colon y \in [0,a]\}$ if $a \in (0,\infty]$, $x \neq \infty$, and $\operatorname{Orb}((\infty,0,\infty) = \{(\infty,0,\infty)\}$, $\ker \operatorname{Ind} \mu = \{0\}$ for $\mu = \delta_0 \times \delta_0 \times m$ as the projection of any open invariant subset U on the third coordinate contains always an interval. $\square$

Let $\mathcal{K}$ be the C^*-algebra of compact operators acting on a separable Hilbert space. Denote by $C_{\mathrm{b}}(\mathbb{R}^+, \mathcal{K})$ the C^*-algebra of bounded complex-valued continuous functions $f : \mathbb{R}^+ \to \mathcal{K}$.

Theorem 3. *There exists an embedding of $C^*(\mathcal{G})$ into $C_{\mathrm{b}}((0,+\infty), \mathcal{K})$ and hence $C^*(\mathcal{G})$ is postliminal.*

Proof. We will construct a homomorphism from the subalgebra $\mathcal{A}$ from Proposition 1 to $C_{\mathrm{b}}((0,+\infty), \mathcal{K})$. Let $f \in C_{\mathrm{c}}(\mathbb{R} \times (0,\infty])$. Then $W_a(\tilde{f}) = \int_{-\infty}^{+\infty} f(t,a) U_t^a \, dt$, where $(U_t^a \xi)(x) = \chi_{[t,a]}(x)\xi(x-t)$ for $t > 0$, $U_t^a = (U_{-t}^a)^*$ for $t < 0$ and $U_0^a = I$. We have shown that each representation W_a is unitarily equivalent to $\operatorname{Ind} \delta_{(0,0,a)}$. Moreover, $W_a(\tilde{f})$ is compact. Indeed,

$$\bigl(W_a(\tilde{f})\xi\bigr)(x) = \int_{x-a}^{x} f(t,a)\xi(x-t)\,dt = \int_0^a f(x-s,a)\xi(s)\,ds, \qquad \xi \in L_2\bigl([0,a]\bigr),$$

which is clearly a Hilbert–Schmidt operator.

Define $V_a : L_2([0,1]) \to L_2([0,a])$ as follows:

$$(V_a\xi)(x) = \xi\left(\frac{x}{a}\right).$$

Then

$$\|V_a\xi\|^2 = \int_0^a \left|\xi\left(\frac{x}{a}\right)\right|^2 dx = a\int_0^1 \bigl|\xi(y)\bigr|^2\,dy = a\|\xi\|^2$$

and $S_a = \frac{1}{\sqrt{a}} V_a$ is a unitary operator with $(S_a^*\eta)(x) = \sqrt{a}\,\eta(xa)$, $\eta \in L_2([0,a])$. Moreover,

$$\begin{aligned}\bigl(V_t^a\xi\bigr)(x) &:= \bigl(S_a^* U_t^a S_a \xi\bigr)(x) = U_t^a V_a \xi(ax) = \chi_{[t,a]}(ax)(V_a\xi)(ax-t)\\ &= \chi_{[t,a]}(ax)\xi\bigl((ax-t)/a\bigr) = \chi_{[t/a,1]}(x)\xi(x-t/a) = \bigl(U_{t/a}^1\xi\bigr)(x).\end{aligned}$$

Define now for each $a > 0$ a compact operator $F(a) = S_a^* W_a(\tilde{f}) S_a$. Then $F : (0,\infty) \to \mathcal{K}$ is continuous and $F(a) \to 0$ as $a \to 0$. In fact,

$$S_a^* W_a(\tilde{f}) S_a = \int_{-\infty}^{+\infty} f(t,a) V_t^a\,dt = \int_{-\infty}^{+\infty} f(t,a) U_{t/a}^1\,dt = a\int_{-\infty}^{+\infty} f(as,a) U_s^1\,ds$$

and

$$\begin{aligned}\|F(a)-F(b)\| &= \left\| a\int_{-\infty}^{+\infty} f(as,a)U_s^1\,ds - b\int_{-\infty}^{+\infty} f(bs,b)U_s^1\,ds \right\| \\ &\leqslant \int_{-1}^{1} \bigl|af(as,a)-bf(bs,b)\bigr|\,ds.\end{aligned}$$

As f is continuous and compactly supported, $\|F(a)-F(b)\| \to 0$ as $a \to b$.

$$\|F(a)\| \leqslant \int_{-1}^{1} \bigl|af(as,a)\bigr|\,ds = \int_{-a}^{a} \bigl|f(s,a)\bigr|\,ds \to 0 \quad \text{as } a \to 0.$$

Moreover,

$$F(a)\cdot G(a) = S_a^* W_a(\tilde f) S_a \cdot S_a^* W_a(\tilde g) S_a = S_a^* W_a(\tilde f * \tilde g) S_a.$$

Define a homomorphism ψ from the algebra generated by $\tilde f$, $f \in C_c(\mathbb{R} \times (0,\infty])$, to $C_0((0,\infty),\mathcal{K})$ by

$$\psi(g)(a) = S_a^* W_a(g) S_a.$$

Then $\|\psi(g)\| = \sup_a \|\psi(g)(a)\| = \sup_a \|W_a(g)\| \leqslant \|g\|$. Thus ψ can be extended to the whole C^*-algebra $C^*(\mathcal{G})$ by continuity. We have that ψ is injective since $W_a(f)=0$ for any a if and only if $f=0$ as the representation $\int_{a>0}^{\oplus} \operatorname{Ind}\delta_{(0,0,a)}\,dm$ is faithful by Proposition 2. □

Our aim now will be to show that all the representations of $C^*(\mathcal{G})$ come from the semigroups of partial isometries.

Let $U = \mathcal{G}^0 \setminus \{(\infty,0,\infty)\}$, which is an open invariant subset in $\mathcal{G}^0$. Let J be the closure of $C_c(\mathcal{G}|_U) = \{f \in C_c(\mathcal{G} \setminus \{(\infty,s,\infty)\colon s \in \mathbb{R}\})\}$. We have $C_c(\mathcal{G}|_U) \subset \{f \in C_c(\mathcal{G})\colon (\operatorname{supp} f)_1 \subset [0,C]$ for some $C>0\}$. By [14, Proposition 4.4], J is a two-sided ideal of $C^*(\mathcal{G})$ and $C^*(\mathcal{G})/J \simeq C^*(\mathcal{G}|_F)$, where $F = \mathcal{G}^0 \setminus U$. Clearly $\mathcal{G}|_F \simeq \mathbb{R}$ so that $C^*(\mathcal{G}|_F) \simeq C^*(\mathbb{R}) \simeq C_0(\mathbb{R})$.

Theorem 4. *The ideal J is isomorphic to $C_0((0,\infty]) \otimes \mathcal{K}$ and $C^*(\mathcal{G})/J \simeq C_0(\mathbb{R})$.*

Thus we have an exact sequence

$$0 \to C_0\bigl((0,\infty]\bigr) \otimes \mathcal{K} \to C^*(\mathcal{G}) \to C_0(\mathbb{R}) \to 0.$$

Proof. Let H_a be the Hilbert space $L_2([0,a])$ which we consider as a subspace of $L_2([0,\infty))$ and let

$$\mathcal{B} = \bigl\{f \in C_0\bigl((0,\infty]\bigr) \otimes \mathcal{K}\colon\ f(a)H_a \subset H_a,\ f(a)|_{H_a^\perp} = 0\bigr\}.$$

By Proposition A.1, $\mathcal{B} \simeq C_0((0,\infty]) \otimes \mathcal{K}$. Thus it suffices to construct an isomorphism between J and the algebra $\mathcal{B}$.

For $f \in C_c(\mathcal{G}|_U)$ let $\psi(f)(a) = \widetilde{W_a}(f)$, where

$$\big(\widetilde{W_a}(f)\xi\big)(x) = \chi_{[0,a]}(x) \int_{x-a}^{x} f(x,t,a)\xi(x-t)\,dt = \int_0^a \chi_{[0,a]}(x) f(x,x-t,a)\xi(t)\,dt,$$

$\xi \in L_2([0,\infty))$. If P_a denotes the projection from $L_2([0,\infty))$ onto $L_2([0.a])$, then $\widetilde{W_a}(f) = W_a(f)P_a$. If a is finite then clearly $\widetilde{W_a}(f)$ is a Hilbert–Schmidt and therefore compact operator such that $\widetilde{W_a}(f)H_a \subset H_a$ and $\widetilde{W_a}(f)|_{H_a^\perp} = 0$.

Our aim now is to prove that ψ can be extended to an isomorphism $\psi : J \to \mathcal{B}$. For this we show first that $\psi(f)$ is continuous whenever $f \in J$. We start with proving the convergence $\widetilde{W_a}(f) \to \widetilde{W_\infty}(f)$ as $a \to \infty$. Assume that $(\operatorname{supp} f)_1 \subset [0,C]$, $(\operatorname{supp} f)_2 \subset [-A,A]$, $A, C > 0$, and let $a > A + C$. Then

$$\big\|\widetilde{W_a}(f)\xi - \widetilde{W_\infty}(f)\xi\big\| = \left(\int_0^{+\infty}\left|\int_0^a \big(f(x,x-t,\infty) - \chi_{[0,a]}(x)f(x,x-t,a)\big)\xi(t)\,dt \right.\right.$$
$$\left.\left. + \int_a^{+\infty} f(x,x-t,\infty)\,dt\right|^2 dx\right)^{1/2} \leqslant I_1 + I_2,$$

where

$$I_1 = \left(\int_0^{+\infty}\left|\int_0^a \big(f(x,x-t,\infty) - \chi_{[0,a]}(x)f(x,x-t,a)\big)\xi(t)\,dt\right|^2 dx\right)^{1/2}$$

and

$$I_2 = \left(\int_0^{+\infty}\left|\int_a^{+\infty} f(x,x-t,\infty)\,dt\right|^2 dx\right)^{1/2}.$$

As $(\operatorname{supp} f)_1 \subset [0,C] \subset [0,a-A]$, we have $f(x,x-t,\infty) = 0$ if $x > a-A$. On the other hand, $(\operatorname{supp} f)_2 \subset [-A,A]$ so that $\chi_{[a,+\infty)}(t)f(x,x-t,\infty) = 0$ if $x - t < -A$ and in particular if $x < a - A$. Thus $I_2 = 0$. Using the above condition on the support of f we obtain also that

$$I_1^2 \leqslant \left(\int_0^C \int_0^{C+A} \big|f(x,x-t,\infty) - f(x,x-t,a)\big|^2\,dt\,dx\right)\|\xi\|^2.$$

Thus $\|\widetilde{W_a}(f) - \widetilde{W_\infty}(f)\|^2 \leqslant \int_0^C \int_0^{C+A} |f(x,x-t,\infty) - f(x,x-t,a)|^2\,dt\,dx$ and since $f(x,x-t,a) \to f(x,x-t,\infty)$, as $a \to \infty$, uniformly with respect to (x,t) on each compact set, we obtain $\|\widetilde{W_a}(f) - \widetilde{W_\infty}(f)\| \to 0$ as $a \to \infty$. This gives the continuity of $\psi(f)$ at $a = \infty$ and compactness of $\psi(f)(\infty)$. In a similar way one proves the continuity of $\psi(f)$ at $a < \infty$. As $\|\widetilde{W_a}(f)\| = \|W_a(f)\|$ which tends to 0 as $a \to 0$ (see the proof of Theorem 3) we have now that $\psi(f) \in \mathcal{B}$.

Clearly, ψ is a $*$-homomorphism from $C_c(\mathcal{G}|_U)$ to $\mathcal{B}$ and $\|\psi(f)(a)\| \leqslant \|f\|_{C^*(\mathcal{G})}$ for each $f \in C_c(\mathcal{G}|_U)$ and $a \in (0, \infty]$. Thus ψ can be extended to a $*$-homomorphism from $C^*(\mathcal{G}|_U)$ to $\mathcal{B}$ denoted by the same letter. We have that ψ is injective since $\widetilde{W_a}(f) = 0$ for all a if and only if $f = 0$. The latter follows from the faithfulness of the representation $\int_{a>0}^{\oplus} \operatorname{Ind} \delta_{(0,0,a)}\, dm$ and the unitary equivalence of W_a and $\operatorname{Ind} \delta_{(0,0,a)}$ for all $a > 0$.

It is left to show now that ψ is surjective. As $\mathcal{B}$ is postliminal it is enough to prove that $\psi(C^*(\mathcal{G}|_U))$ is a rich subalgebra of $\mathcal{B}$ [6, 11.1]. Any irreducible representation of $\mathcal{B}$ is unitarily equivalent to $\pi_t' = \pi_t|_{V_t}$, where $\pi_t(f) = f(t)$, $t \in (0, \infty]$ (see the proof of Proposition A.1). Then $\pi_t'(\psi(f)) = W_t(f)$ for all $f \in C^*(\mathcal{G}|_U)$. Since $\pi_t' \circ \psi|_{C^*(\mathcal{G}|_U)} = W_t|_{C^*(\mathcal{G}|_U)}$ is an irreducible representation of $C^*(\mathcal{G}|_U)$ as the restriction of the irreducible representation W_t of $C^*(\mathcal{G})$ we obtain that $\pi_t'|_{\psi(C^*(\mathcal{G}|_U))}$ is irreducible. Further if π_1, π_2 are non-equivalent irreducible representations of $\mathcal{B}$ then $\pi_1 = \pi_t'$, $\pi_2 = \pi_s'$, for $t \neq s$ up to a unitary equivalence. As for different t, W_t are non-equivalent representations of $C^*(\mathcal{G})$ and hence of the ideal $C^*(\mathcal{G}|_U)$ we obtain that $\pi_1|_{\psi(C^*(\mathcal{G}|_U))}$ and $\pi_2|_{\psi(C^*(\mathcal{G}|_U))}$ are non-equivalent. Thus $\psi(C^*(\mathcal{G}|_U))$ is a rich subalgebra of $\mathcal{B}$ and the proof is complete. □

As a corollary we get the following.

Theorem 5. *Any irreducible representation of $C^*(\mathcal{G})$ is unitarily equivalent to some W_a, $a \in (0, +\infty]$, or ω_ϕ, $\phi \in \mathbb{R}$.*

Proof. It follows from the proof of Theorem 4 that any irreducible representation of J is unitarily equivalent to some $W_a|_J$, $a \in (0, +\infty]$. Hence any irreducible representation of $C^*(\mathcal{G})$ is either unitarily equivalent to W_a, $a \in (0, +\infty]$ or to an irreducible representation which is a composition of an irreducible representation of the quotient algebra $C^*(\mathcal{G})/J \simeq C_0(\mathbb{R})$ and the canonical homomorphism from $C^*(\mathcal{G})$ to $C^*(\mathcal{G})/J$ and thus to a one-dimensional representation ω_ϕ, $\phi \in \mathbb{R}$. □

Theorem 6. *Any representation of $C^*(\mathcal{G})$ is unitarily equivalent to $\bigoplus_{n=1}^{\infty} \operatorname{Ind} \mu_n \otimes I_n \oplus \int_{\mathbb{R}} \omega_\phi \otimes I_n\, d\nu_n$ for some Radon measures μ_n on $\mathcal{G}^0$ and ν_n on $\mathbb{R}$.*

Proof. By Theorem 5 the spectrum of $\mathcal{A} = C^*(\mathcal{G})$ as a set is equal to $\{W_a\}_{a \in (0,\infty]} \cup \{\omega_\phi\}_{\phi \in \mathbb{R}}$. It is homeomorphic to a topological space $X = (0, \infty] \sqcup_\gamma \mathbb{R}$: here X is a disjoint union of $(0, \infty]$ and $\mathbb{R}$, $\gamma : \mathbb{R} \to (0, \infty]$ is a map given by the rule $\gamma(x) = \infty$ for all $x \in \mathbb{R}$, open sets of the topology on X are the open sets U in $\mathbb{R}$ and the sets $V \sqcup \gamma^{-1}(V)$, where V runs over open sets in $(0, \infty]$. The σ-algebra of Borel sets of X is the same as the σ-algebra of Borel sets of the disjoint union $(0, \infty] \sqcup \mathbb{R}$.

Since by Theorem 2, $C^*(\mathcal{G})$ is a postliminal C^*-algebra, any its representation π has a unique decomposition in a direct sum of representations of a fixed multiplicity n, i.e. $\pi = \bigoplus_{n \in \mathbb{N}} \pi_n$. Whereas each π_n is a sum of n copies of multiplicity free representation $\pi^{(n)}$ which is of the form $\int_{\hat{\mathcal{A}}}^{\oplus} \zeta\, d\mu^{(n)}(\zeta)$ for some Radon measure $\mu^{(n)}$. In our case we can rewrite the last integral as

$$\int_{(0,\infty]}^{\oplus} W_a\, d\mu^{(n)}(a) \oplus \int_{\mathbb{R}}^{\oplus} \omega_\phi\, d\nu^{(n)}(\phi).$$

If we let μ_n be the measure on $\mathcal{G}^0$ induced from $\mu^{(n)}$ via the map $\rho : \mathcal{G}^0 \to (0,\infty]$, $(x,0,a) \mapsto a$ and let $\nu_n = \nu^{(n)}$ then we obtain the desired decomposition. $\square$

Let $\mathrm{Rep}(U, H)$ denote the category of strongly-continuous one-parameter semigroups of partial isometries acting on a Hilbert space H and $\mathrm{Rep}(C^*(\mathcal{G}), H)$ is the category of nondegenerate representations of $C^*(\mathcal{G})$ on H.

If $\{U_t\}_{t\geqslant 0} \in \mathrm{Rep}(U, H)$, then by [8] there exists a unitary S in $B(H)$ such that

$$SU_tS^* = \int_{(0,\infty)}^{\oplus} U_t^{(a)} \otimes I_a \, d\mu(a) \oplus U_t^{\infty} \otimes I_{\infty} \oplus U_t^{\infty *} \otimes I_{-\infty} \oplus \int_{\mathbb{R}}^{\oplus} e^{it\lambda} \otimes I_\lambda \, d\nu(\lambda), \tag{6}$$

here I_a, I_λ, I_∞ and $I_{-\infty}$ are the identity operators acting on some Hilbert spaces H_a, H_λ, H_∞ and $H_{-\infty}$. We permit H_∞ and $H_{-\infty}$ to be zero-dimensional; in this case the corresponding summand $U_t^\infty \otimes I_\infty$ or $U_t^{\infty *} \otimes I_{-\infty}$ will be omitted. Let $\mathrm{Rep}_+(U, H) = \mathrm{Rep}(U, H)/\mathbb{Z}_2$. The elements of $\mathrm{Rep}_+(U, H)$ can be identified with those semigroups $\{U_t\}_{t\geqslant 0}$ for which decomposition (6) holds without the components $U_t^{\infty *} \otimes I_{-\infty}$. For such $\{U_t\}_{t\geqslant 0} \in \mathrm{Rep}_+(U, H)$ we associate a representation $F(\{U_t\})$ of $C^*(\mathcal{G})$ of the form

$$S^* \int_{(0,\infty)}^{\oplus} W_a \otimes I_a \, d\mu(a) \oplus W_\infty \otimes I_\infty \oplus \int_{\mathbb{R}}^{\oplus} \omega_\lambda \otimes I_\lambda \, d\nu(\lambda) S.$$

Corollary 1. *The mapping $F : \mathrm{Rep}_+(U, H) \to \mathrm{Rep}(C^*(\mathcal{G}), H)$ is well defined.*

This follows from the fact that operator S in decomposition (6) of U_t is defined uniquely up to multiplication by a diagonal unitary operator

$$V = \int_{(0,\infty)}^{\oplus} V_a \otimes I_a \, d\mu(a) \oplus V_\infty \otimes I_\infty \oplus V_{-\infty} \otimes I_{-\infty} \oplus \int_{\mathbb{R}}^{\oplus} v_\lambda \otimes I_\lambda \, d\nu(\lambda),$$

where V_a, V_∞, $V_{-\infty}$ are scalar operators.

We will prove in the next section that this mapping is in fact a bijection.

5. Unbounded generator of $C^*(\mathcal{G})$

Let U_t be a strongly continuous semigroup of partial isometries on H. Let us recall that for such a semi-group the (infinitesimal) generator A is defined by

$$A\varphi = \lim_{t\to 0} \frac{1}{t}(U_t - I)\varphi$$

for all φ whenever this limit exists. It is known that A is a closed operator with domain $\mathcal{D}(A)$ which is dense in H and A determines the semigroup uniquely (see [11]).

In this section we construct an unbounded element A which generates $C^*(\mathcal{G})$ as affiliated element (in the sense of [15,16]) and such that for each non-degenerate representation π of $C^*(\mathcal{G})$, $\pi(A)$ is the generator of the corresponding semigroup $\{U_t\}_{t\geqslant 0}$ of partial isometries.

First we recall some important definitions and results from [15,16]. Let $\mathcal{A}$ be a C^*-algebra and T be a linear mapping acting on $\mathcal{A}$ defined on a dense linear domain $D(T)$. The adjoint mapping is defined by the equivalence ($x, y \in \mathcal{A}$):

$$x \in D(T^*) \quad \text{and} \quad y = T^*x \quad \Leftrightarrow \quad \text{for any } a \in D(T) \quad (Ta)^*x = a^*y.$$

Let $B(\mathcal{A})$ be the algebra of bounded linear mappings acting on $\mathcal{A}$. Then $b \in B(\mathcal{A})$ is the Hermitean adjoint of $a \in B(\mathcal{A})$ and write $b = a^*$ if

$$y^*(ax) = (by)^*x$$

for any $x, y \in \mathcal{A}$. The set

$$M(\mathcal{A}) = \{a \in B(\mathcal{A}):\ a^* \in B(\mathcal{A})\}$$

endowed with natural algebraic operations and the sup-norm is the multiplier algebra of $\mathcal{A}$. Assuming that elements of $\mathcal{A}$ act on $\mathcal{A}$ by left multiplication we have the embedding $\mathcal{A} \hookrightarrow M(\mathcal{A})$.

Let T be a linear mapping acting on $\mathcal{A}$ having a dense domain $D(T)$. We say that T is *affiliated* with $\mathcal{A}$ and write $T\eta\mathcal{A}$ if there exists $z \in M(\mathcal{A})$ such that $\|z\| \leqslant 1$, $D(T) = \sqrt{I - z^*z}\mathcal{A}$ and $T\sqrt{I - z^*z}a = za$ for all $a \in \mathcal{A}$. It is known that z is determined by T and one calls it z-transform of T and denotes it by z_T. One has that $T\eta\mathcal{A}$ implies $T^*\eta\mathcal{A}$ and $z_{T^*} = z_T^*$. If $\mathcal{A}$ is a C^*-algebra of operators on H and T is a closed densely defined operator T on H then

$$z_T = T(I + T^*T)^{-1}.$$

The affiliation relation in the theory of C^*-algebras was introduced by S. Baaj and P. Jungl in [2]. This new reference will be under number 2, see References. Let $\operatorname{Rep}(\mathcal{A}, H)$ denote the set of all non-degenerate representations of $\mathcal{A}$ in a Hilbert space H, i.e. all $*$-algebra homomorphisms $\pi : \mathcal{A} \to B(H)$ such that $\pi(\mathcal{A})H$ is dense in H. Each such $\pi \in \operatorname{Rep}\mathcal{A}$ admits the unique extension to a unital $*$-algebra homomorphism $\pi : M(\mathcal{A}) \to B(H)$. If $T\eta\mathcal{A}$ and $\pi \in \operatorname{Rep}(\mathcal{A}, H)$ then there exists a uniquely defined closed operator S acting on H such that $\pi(z_T) = z_S$. We write $\pi(T) = S$.

For a C^*-algebra $\mathcal{B}$ of operators on H we set

$$\operatorname{Mor}(\mathcal{A}, \mathcal{B}) = \{\pi \in \operatorname{Rep}(\mathcal{A}, H):\ \overline{\pi(\mathcal{A})\mathcal{B}} = \mathcal{B}\}.$$

Let $T_1, \ldots, T_n$ be elements affiliated with $\mathcal{A}$. One says that $\mathcal{A}$ is generated by $T_1, \ldots, T_n$ if for any Hilbert space H, any C^*-algebra $\mathcal{B}$ of operators in $B(H)$ and any $\pi \in \operatorname{Rep}(\mathcal{A}, H)$ one has

$$\pi(T_i)\eta\mathcal{B} \quad \text{for any } i = 1, \ldots, n \quad \Rightarrow \quad \pi \in \operatorname{Mor}(\mathcal{A}, \mathcal{B}).$$

In order to find a generator of $C^*(\mathcal{G})$ we will need to prove several lemmas bellow.

For the representation W_a of $C^*(\mathcal{G})$ we denote by $W_a(A)$ the infinitesimal operator of the corresponding semigroup. By [12], we have that $\mathcal{D}(W_a(A))$ consists of all absolutely continuous functions $f \in L_2([0, a])$, if $a \neq \infty$, and $f \in L_2([0, \infty))$, if $a = \infty$, satisfying $f(0) = 0$, and $(W_a(A)f)(x) = -f'(x)$, $f \in \mathcal{D}(W_a(A))$.

Let

$$h(x,t,a)=\begin{cases}-\chi_{[0,x]}(t)\sinh t+\frac{\sinh x}{\cosh a}\cosh(x-t-a), & a<\infty,\\ -\chi_{[0,x]}(t)\sinh t+\sinh x e^{t-x}, & a=\infty, x\neq\infty,\\ e^{-|t|}/2, & a=\infty, x=\infty,\end{cases}$$

$$\tilde{h}(x,t,a)=\begin{cases}\chi_{[0,x]}(t)\cosh t-\frac{\cosh(x-t-a)\cosh x}{\cosh a}, & a<\infty,\\ \chi_{[0,x]}(t)\cosh t-\cosh x e^{t-x}, & a=\infty, x\neq\infty,\\ -e^{-|t|}/2, & a=\infty, x=\infty,\end{cases}$$

$$k(x,t,a)=\begin{cases}-\chi_{[0,x]}(t)\sinh t-\dfrac{\cosh x}{\cosh a}\sinh(x-t-a), & a<\infty,\\ -\chi_{[0,x]}(t)\sinh t+\cosh x e^{t-x}, & a=\infty, x\neq\infty,\\ e^{-|t|}/2, & a=\infty, x=\infty,\end{cases}$$

$$\tilde{k}(x,t,a)=\begin{cases}-\chi_{[0,x]}(t)\cosh t-\frac{\sinh(x-t-a)\sinh x}{\cosh a}, & a<\infty,\\ -\chi_{[0,x]}(t)\cosh t+\sinh x e^{t-x}, & a=\infty, x\neq\infty,\\ -e^{-|t|}/2, & a=\infty, x=\infty.\end{cases}$$

Lemma 1. *The elements h, $\tilde{h}$, k, $\tilde{k}$ belong to $C^*(\mathcal{G})$. Moreover,*

$$W_a(h)=\bigl(1+W_a(A)^*W_a(A)\bigr)^{-1},\quad W_a(\tilde{h})=W_a(A)\bigl(1+W_a(A)^*W_a(A)\bigr)^{-1},$$
$$W_a(k)=\bigl(1+W_a(A)W_a(A)^*\bigr)^{-1},\quad W_a(\tilde{k})=W_a(A)^*\bigl(1+W_a(A)W_a(A)^*\bigr)^{-1},$$

for any $a\in(0,\infty]$.

Proof. To show the first statement it suffices to see that $\int_{x-a}^{x}|h(x,t,a)|\,dt$, $\int_{x-a}^{x}|\tilde{h}(x-t,-t,a)|\,dt$, $\int_{x-a}^{x}|\tilde{h}(x,t,a)|\,dt$ and $\int_{x-a}^{x}|\tilde{h}(x-t,-t,a)|\,dt$ are uniformly bounded on $\{(x,a)\colon x\in[0,a], a>0\}$. We have that

$$\int_{x-a}^{x}\bigl|h(x,t,a)\bigr|\,dt=\frac{2\cosh(a-x)\sinh^2(x/2)}{\cosh a}+\frac{\sinh x\sinh(a-x)}{\cosh a}$$
$$\leqslant\frac{\cosh a-1}{\cosh a}+\frac{\cosh a-1}{2\cosh a}\leqslant\frac{3}{2}.$$

Similar calculations give us the uniform boundedness of $\int_{x-a}^{x}|\tilde{h}(x,t,a)|\,dt$.

Then $(W_a(A)^*W_a(A)f)(x)=-f''(x)$ with the domain consisting of $f\in L_2([0,a])$ such that f and f' are absolutely continuous, $f', f''\in L_2([0,a])$ satisfy the boundary conditions $f(0)=0$ and $f'(a)=0$ if $0\leqslant a<\infty$ and $f(0)=0$ if $a=+\infty$.

Let $f,g\in L^2([0,a])$, $f\in\mathcal{D}(W_a(A)^*W_a(A))$ and $((1+W_a(A)^*W_a(A))f)(x)=-g(x)$. Then $f''(x)-f(x)=g(x)$ with general solution given by

$$f(x)=\bigl(e^x-e^{-x}\bigr)h_1+\bigl(e^x+e^{-x}\bigr)h_2+\frac{1}{2}\int_0^x e^{-(t-x)}g(t)\,dt-\frac{1}{2}\int_0^x e^{t-x}g(t)\,dt, \tag{7}$$

where h_1, h_2 are constants depending on g. Let first $0 < a < +\infty$. Since $f \in \mathcal{D}(W_a(A)^* W_a(A))$, $f(0) = 0$ and $f'(a) = 0$ which imply $h_2 = 0$ and $h_1 = -\frac{1}{2\cosh a}\int_0^a \cosh(t-a)g(t)\,dt$.

Thus

$$\begin{aligned}
&\big(\big(\big(1 + W_a(A)^* W_a(A)\big)^{-1}\big)g\big)(x)\\
&= \frac{\sinh x}{\cosh a}\int_0^a \cosh(t-a)g(t)\,dt - \int_0^x \sinh(x-t)g(t)\,dt\\
&= \frac{\sinh x}{\cosh a}\int_{x-a}^x \cosh(x-t-a)g(x-t)\,dt - \int_{x-a}^x \chi_{[0,x]}(t)\sinh t\, g(x-t)\,dt\\
&= \int_{\mathbb{R}} h(x,t,a)\big(U_t^a g\big)(x)\,dt = \big(W_a(h)g\big)(x),
\end{aligned}$$

where

$$h(x,t,a) = -\chi_{[0,x]}(t)\sinh t + \frac{\sinh x}{\cosh a}\cosh(x-t-a).$$

If $a = \infty$, one chooses $h_2 = 0$ and $h_1 = -\frac{1}{2}\int_0^\infty e^{-t} g(t)\,dt$ and obtains the equality $(1 + W_a(A)^* W_a(A))^{-1} = W_a(h)$. Next we calculate $W_a(A)(1 + W_a(A)^* W_a(A))^{-1}$. For $0 < a < +\infty$,

$$\begin{aligned}
&\big(W_a(A)\big(1 + W_a(A)^* W_a(A)\big)^{-1} g\big)(x)\\
&= -\left(\frac{\sinh x}{\cosh a}\int_0^a \cosh(t-a)g(t)\,dt - \int_0^x \sinh(x-t)g(t)\,dt\right)'\\
&= -\frac{\cosh x}{\cosh a}\int_0^a \cosh(t-a)g(t)\,dt + \int_0^x \cosh(x-t)g(t)\,dt\\
&= \int_{\mathbb{R}} \tilde{h}(x,t,a)\big(U_t^a g\big)(x)\,dt = \big(W_a(\tilde{h})g\big)(x).
\end{aligned}$$

In a similar way one checks the case $a = \infty$. To show the statement about k, and $\tilde{k}$ take $f, g \in L^2([0,a])$, $f \in \mathcal{D}(W_a(A)W_a(A)^*)$ and $((1 + W_a(A)W_a(A)^*)f)(x) = -g(x)$. Then $f''(x) - f(x) = g(x)$ as in the previous case with general solution given by (7). If $0 < a < +\infty$ we have for $f \in \mathcal{D}(W_a(A)W_a(A)^*)$ that $f(a) = 0$ and $f'(0) = 0$ implying

$$h_1 = 0 \quad \text{and} \quad h_2 = \frac{1}{2\cosh a}\int_0^a \sinh(t-a)g(t)\,dt.$$

If $a=+\infty$ one has that any $f\in\mathcal{D}(W_a(A)W_a(A)^*)$ satisfies $f'(0)=0$. To get also a solution $f\in L_2([0,\infty))$ one chooses $h_1=0$ and $h_2=-\frac{1}{2}\int_0^\infty e^{-t}g(t)\,dt$. Similar verification gives

$$\big(\big(1+W_a(A)W_a(A)^*\big)^{-1}g\big)(x)=\int\limits_{\mathbb{R}}k(x,t,a)\big(U_t^a g\big)(x)\,dt=\big(W_a(k)g\big)(x)$$

and

$$\big(W_a(A)^*\big(1+W_a(A)W_a(A)^*\big)^{-1}g\big)(x)=\int\limits_{\mathbb{R}}\tilde{k}(x,t,a)\big(U_t^a g\big)(x)\,dt=\big(W_a(\tilde{k})g\big)(x). \qquad \square$$

As $W_a(h)=(1+W_a(A)^*W_a(A))^{-1}$ is a positive operator for each $a>0$ and $\pi=\int_{a>0}^{\oplus}W_a\,da$ is a faithful representation of $C^*(\mathcal{G})$, h is a positive element of $C^*(\mathcal{G})$. Consider the linear subspace $U=\{h^{1/2}b,\ b\in C^*(\mathcal{G})\}$ (here $h^{1/2}$ is the square root of h in $C^*(\mathcal{G})$) and define an operator z from U to $C^*(\mathcal{G})$ by

$$zh^{1/2}b=\tilde{h}b.$$

Lemma 2. *The subspace U is dense in $C^*(\mathcal{G})$. The mapping z can be extended to an element $z\in M(C^*(\mathcal{G}))$ such that $\|z\|\leqslant 1$ and $\sqrt{1-z^*z}C^*(\mathcal{G})$ is dense in $C^*(\mathcal{G})$.*

Proof. Assume that U is not dense in $C^*(\mathcal{G})$. Then the closure $\overline{U}$ is a proper right ideal. By [6, 2.9.5] there exists an irreducible representation π on H and a cyclic vector $\varphi\in H$ such that $(\pi(h^{1/2}b)\varphi,\varphi)=0$, $b\in C^*(\mathcal{G})$ implying $(\pi(b)\varphi,\pi(h^{1/2})\varphi)=0$ and $\pi(h^{1/2})\varphi=0$. By Theorem 5 any irreducible representation π is unitarily equivalent to one from the set $S=\{W_a,\omega_\phi,a\in\overline{\mathbb{R}^+},\phi\in\mathbb{R}\}$. If π is unitarily equivalent to W_a, $(a\in(0,\infty])$, then $\ker\pi(h^{1/2})=\{0\}$ since $\ker W_a(h^{1/2})=\ker(1+W_a(A)^*W_a(A))^{-1}=\{0\}$. For $\pi=\omega_\phi$ we have $\pi(h)=1/2$ and therefore $\ker\omega_\phi(h^{1/2})$ is also trivial. A contradiction. As

$$\begin{aligned}\|zh^{1/2}b\|&=\sup_{a>0}\|W_a(zh^{1/2}b)\|=\sup_{a\geqslant 0}\|W_a(\tilde{h}b)\|\\&=\sup_{a>0}\big\|\big(W_a(A)\big(1+W_a(A)^*W_a(A)\big)^{-1/2}\big)W_a\big(h^{1/2}\big)W_a(b)\big\|\\&\leqslant\sup_{a>0}\big\|W_a\big(h^{1/2}b\big)\big\|=\|h^{1/2}b\|,\end{aligned}$$

z can be extended to a bounded operator on $\overline{U}=C^*(\mathcal{G})$ with the norm $\|z\|\leqslant 1$. We have also that for $\pi=W_a$ on a Hilbert space H the equality $\pi(z)\pi(h)^{1/2}\pi(b)\varphi=\pi(\tilde{h})\pi(b)\varphi$ is equivalent to

$$W_a(z)\big(1+W_a(A)^*W_a(A)\big)^{-1/2}W_a(b)\varphi=W_a(A)\big(1+W_a(A)^*W_a(A)\big)^{-1}W_a(b)\varphi$$

giving us that $W_a(z)\psi=W_a(A)(1+W_a(A)^*W_a(A))^{-1/2}\psi$ for any $\psi\in H$. Here we denote by π also the extension of a non-degenerate representation π of $C^*(\mathcal{G})$ to the space of bounded linear mappings on $C^*(\mathcal{G})$.

To prove that $z\in M(C^*(\mathcal{G}))$ we have to show that z^* is also bounded. Consider the set $\tilde{U}=\{k^{1/2}a\colon a\in C^*(\mathcal{G})\}$. In a similar way one shows that $\tilde{U}$ is dense in $C^*(\mathcal{G})$. Define an operator

$\tilde{z}$ on $\tilde{U}$ by $\tilde{z}k^{1/2}a = \tilde{k}a$ for all $a \in C^*(\mathcal{G})$. As above one checks that $\tilde{z}$ can be extended to a bounded mapping on $C^*(\mathcal{G})$ with the norm $\|\tilde{z}\| \leqslant 1$. One also has that $W_a(\tilde{z})\psi = W_a(A)^*(1 + W_a(A)W_a(A)^*)^{-1/2}\psi$ for any $\psi \in H$. Our aim now is to show that $z^* = \tilde{z}$. Take $y \in \tilde{U}$, i.e., $y = k^{1/2}a$, $a \in C^*(\mathcal{G})$, and $x \in U$, i.e., $x = h^{1/2}b$, $b \in C^*(\mathcal{G})$. We have to prove the equality $y^*(zx) = (\tilde{z}y)^*x$, i.e., $b^*k^{1/2}\tilde{h}a = b^*\tilde{k}^*h^{1/2}a$. For this it is enough to see that $k^{1/2}\tilde{h} = \tilde{k}^*h^{1/2}$. Applying the representation W_a to the equality we obtain

$$\begin{aligned}&\bigl(1 + W_a(A)W_a(A)^*\bigr)^{-1/2}W_a(A)\bigl(1 + W_a(A)^*W_a(A)\bigr)^{-1}\\ &\quad= \bigl((W_a)(A)^*\bigl(1 + W_a(A)W_a(A)^*\bigr)^{-1}\bigr)^*\bigl(1 + W_a(A)^*W_a(A)\bigr)^{-1/2}.\end{aligned} \tag{8}$$

As $z_T = T(1 + T^*T)^{-1/2}$, we arrive that (8) is equivalent to $z^*_{W_a(A)} = z_{W_a(A)^*}$ which clearly holds.

Direct calculation shows now also that $\pi(\sqrt{1 - z^*z}) = (1 + W_a(A)^*W_a(A))^{-1/2} = \pi(h^{1/2})$. Thus $\sqrt{1 - z^*z} = h^{1/2}$ giving the density of $\sqrt{1 - z^*z}\,C^*(\mathcal{G})$. □

Define now a linear map A acting on $C^*(\mathcal{G})$ with domain $D(A) = \sqrt{1 - z^*z}\,C^*(\mathcal{G})$ as follows: $A\sqrt{1 - z^*z}\,b = zb$ for all $b \in C^*(\mathcal{G})$. A is affiliated with $C^*(\mathcal{G})$ by the construction and $z_A = z$.

Proposition 7. *The mapping* $F : \mathrm{Rep}_+(U, H) \to \mathrm{Rep}(C^*(\mathcal{G}), H)$ *is a bijection.*

Proof. A semigroup $\{U_t\}_{t \geqslant 0} \in \mathrm{Rep}_+(U, H)$ is uniquely defined by its generator $\mathcal{A}$. It is easy to deduce from the definition of F that the z-transform $z_{\mathcal{A}}$ is equal to $\pi(z)$ where π is a unique extension of representation $F(\{U_t\})$ of the C^*-algebra $C^*(\mathcal{G})$ to its multiplier algebra $M(C^*(\mathcal{G}))$ and $z \in M(C^*(\mathcal{G}))$ is the z-transform of the affiliated element A defined above. Since the correspondence $\mathcal{A} \to z_{\mathcal{A}}$ is one-to-one we conclude that F is an injection. That F is a surjection follows from Theorem 6. □

Theorem 8. *The linear mapping A generates $C^*(\mathcal{G})$ as affiliated element. If π is a representation of $C^*(\mathcal{G})$ on H, then $\pi(A)\varphi = A_\pi\varphi$, for all $\varphi \in \pi(D(A))H$, where A_π is the generator of the corresponding semigroup $\{U_t\}_{t \geqslant 0}$ of partial isometries.*

Proof. To prove that A generates the algebra, by [16, Theorem 3.3], it is enough to show that A separate representations of $C^*(\mathcal{G})$, i.e. if π_1, π_2 are different non-degenerate representations of $C^*(\mathcal{G})$ in a Hilbert space H then $\pi_1(A) \neq \pi_2(A)$, and that $(1 + A^*A)^{-1} \in C^*(\mathcal{G})$. The first claim follows from Proposition 7 and the fact that a semigroup is uniquely defined by its generator. The second follows from the equality $(1 + A^*A)^{-1} = 1 - z^*z = h$ obtained in Lemma 2.

For a representation π we have $\pi(A)\pi(\sqrt{1 - z^*z})\pi(b)\varphi = \pi(z)\varphi$. As $\pi(z) = z_{A_\pi} = A_\pi(1 + A_\pi^*A_\pi)^{-1/2}$ and $\pi(\sqrt{1 - z^*z}) = (1 + A_\pi^*A_\pi)^{-1/2}$ we have the required equality between $\pi(A)$ and A_π. □

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Appendix A

Let $\mathcal{K}$ be the algebra of compact operators on $H = L_2([0, +\infty))$ and let $\mathcal{A} = C_0((0, +\infty], \mathcal{K})$. Denote by H_a the Hilbert space $L_2([0, a])$ which we consider as a subspace of H and let

$$\mathcal{B} = \left\{ f \in \mathcal{A}\colon\ f(a)H_a \subset H_a,\ f(a)|_{H_a^\perp} = 0 \right\}.$$

Proposition A.1.

$$\mathcal{B} \simeq \mathcal{A}.$$

Proof. Let us denote by $\mathcal{K}$ also the algebra of all compact operators on $L_2([0, 1])$. Clearly, the algebra $\mathcal{C} = C_0((0, 1], \mathcal{K})$ is isomorphic to the algebra $\mathcal{A}$. Let for $0 < t \leqslant 1$, $V_t = L_2([0, t])$. Considering V_t as subspaces of $L_2([0, 1])$ we let

$$\mathcal{D} = \left\{ g \in \mathcal{C}\colon\ g(t)V_t \subset V_t,\ g(t)|_{V_t^\perp} = 0 \right\}.$$

Proving that $\mathcal{B}$ is isomorphic to $\mathcal{D}$ and then that $\mathcal{D}$ is isomorphic to $\mathcal{C}$ we get the statement of the proposition. Let $S : L_2([0, +\infty)) \to L_2([0, 1])$ be the isometric isomorphism given by $(S\xi)(t) = \frac{1}{1-t}\xi(\frac{t}{1-t})$. We have

$$SH_a = V_{\frac{a}{1+a}}, \quad a > 0. \tag{A.1}$$

Define now a mapping $\varphi : \mathcal{B} \to \mathcal{C}$ by

$$\varphi(f)(a) = Sf\left(\frac{a}{1-a}\right)S^*, \quad a \in (0, 1].$$

By (A.1) $\varphi(f)(a)V_a \subset V_a$ and $\varphi(f)(a)|_{V_a^\perp} = 0$ and therefore $\varphi(\mathcal{B}) \subseteq \mathcal{D}$. One can easily see that φ is even a $*$-isomorphism between $\mathcal{B}$ and $\mathcal{D}$. Define $S_a : V_a \to L_2([0, 1])$ as follows

$$(S_a\xi)(x) = \sqrt{a}\,\xi(ax).$$

Then S_a is a unitary operator. For $f \in \mathcal{D}$ we set

$$\psi(f)(a) = S_a f(a) S_a^*.$$

For each $a \in (0, 1]$, $\psi(f)(a) \in \mathcal{K}$. To prove that $\psi(f) \in \mathcal{C}$ we have to show that the mapping $a \mapsto \psi(f)(a)$ is continuous. We check this first for $f = T_{\eta,\zeta}$, where

$$\eta, \zeta \in \mathcal{X} := \left\{ \xi \in C\big((0, 1], C[0, 1]\big)\colon\ \xi_a \in V_a,\ a \in (0, 1] \right\} \quad \text{and}$$
$$\big(T_{\eta,\zeta}(a)\xi\big)(x) = (T_{\eta_a,\zeta_a}\xi)(x) = (\xi, \eta_a)\zeta_a(x).$$

We have

$$S_a T_{\eta_a,\zeta_a} S_a^* \xi = \big(S_a^*\xi, \eta_a\big)S_a\zeta_a = (\xi, S_a\eta_a)S_a\zeta_a$$

and for $a, b \in (0, 1]$

$$\begin{aligned}\|S_a T_{\eta_a,\zeta_a} S_a^* \xi - S_b T_{\eta_b,\zeta_b} S_b^* \xi\| &= \|(\xi, S_a \eta_a) S_a \zeta_a - (\xi, S_b \eta_b) S_b \zeta_b\| \\ &\leqslant \|(\xi, S_a \eta_a - S_b \eta_b) S_a \zeta_a\| + |(\xi, S_b \eta_b)| \cdot \|S_a \zeta_a - S_b \zeta_b\| \\ &\leqslant \|S_a \eta_a - S_b \eta_b\| \cdot \|\zeta_a\| \cdot \|\xi\| + \|S_a \zeta_a - S_b \zeta_b\| \cdot \|\eta_b\| \cdot \|\xi\|\end{aligned}$$

so that

$$\|S_a T_{\eta_a,\zeta_a} S_a^* - S_b T_{\eta_b,\zeta_b} S_b^*\| \leqslant \|S_a \eta_a - S_b \eta_b\| \cdot \|\zeta_a\| + \|S_a \zeta_a - S_b \zeta_b\| \cdot \|\eta_b\|.$$

Further

$$\|S_a \eta_a - S_b \eta_b\|^2 = \int_0^1 \left|\sqrt{a}\eta_a(ax) - \sqrt{b}\,\eta_b(bx)\right|^2 dx \to 0 \quad \text{as } a \to b$$

since $\eta(a, x) = \eta_a(ax)$ is continuous. Similarly, $\|S_a \zeta_a - S_b \zeta_b\| \to 0$ as $a \to b$. Moreover, $\|\zeta_a\|$ is bounded in a neighborhood of b giving $\|S_a T_{\eta_a,\zeta_a} S_a^* - S_b T_{\eta_b,\zeta_b} S_b^*\| \to 0$ as $a \to b$, i.e. $\psi(T_{\eta,\zeta})$ is continuous.

Next we prove that the linear span, $\mathcal{M}$, of elements $T_{\eta,\zeta}$ is dense in $\mathcal{D}$. Since $\mathcal{D}$ is postliminal, it is enough to prove that $\mathcal{M}$ is rich [6, 11.1]. Note first that $\mathcal{D}$ is a subalgebra of $\mathcal{C}$ and since any irreducible representation of $\mathcal{C}$ is unitary equivalent to $\pi_t(f) = f(t)$, $f \in \mathcal{C}$ one can easily obtain that any irreducible representation of $\mathcal{D}$ is given by $\pi_t' : f \mapsto f(t)|_{V_t}$ up to unitary equivalence. For any fixed $t \in (0, 1]$ we have $\pi_t'(T_{\eta,\zeta}) = T_{\eta_t,\zeta_t}$. Moreover, for any $\eta', \zeta' \in C([0, t]) \subset L_2([0, t])$ there exist $\eta, \zeta \in \mathcal{X}$ such that $\eta_t = \eta'$ and $\zeta_t = \zeta'$. For example, one can take $\eta_t = S_t^* \eta'$, $\zeta_t = S_t^* \zeta'$. Hence $\pi_t(\mathcal{M})$ is dense in the algebra of compact operators on $L_2([0, t])$ ensuring that π_t' is irreducible as representation of $\mathcal{M}$. Next we claim that for any pair of non-equivalent irreducible representations π_1, π_2 of $\mathcal{D}$ (up to a unitary equivalence we have that $\pi_1 = \pi_t'$, $\pi_2 = \pi_s'$ for distinct $t, s \in (0, 1]$) the restrictions $\pi_1|_{\mathcal{M}}$, $\pi_2|_{\mathcal{M}}$ are also non-equivalent. For this it is enough to see that there is no unitary operator $U : L_2([0, s]) \to L_2([0, t])$ such that $U T_{\eta_s,\zeta_s} U^* = T_{\eta_t,\zeta_t}$, i.e. $U \zeta_s = \zeta_t$, $U \eta_s = \eta_t$, for any $\eta, \zeta \in \mathcal{X}$. Take $\eta \in C([0, 1])$ such that $\operatorname{supp} \eta = [0, 1]$. Assume $t < s$ and take arbitrary $b \in (t, s)$. Set

$$\eta_x = S_x^* \eta \quad \text{and} \quad \zeta_x = \begin{cases} S_x^* \eta, & x < b, \\ S_b^* \eta, & x \geqslant b. \end{cases}$$

It is easy to check that $\eta, \zeta \in \mathcal{X}$. Clearly $\eta_t = \zeta_t$ but since $\operatorname{supp} \eta_s = [0, s] \neq \operatorname{supp} \zeta_s = [0, b]$ we obtain $\eta_s \neq \zeta_s$ which gives the claim. Therefore $\mathcal{M}$ is rich and consequently dense in $\mathcal{D}$ giving $\psi(\mathcal{D}) \subset \mathcal{C}$.

Using similar argument one can obtain that $\psi(\mathcal{D})$ is a rich subalgebra of $\mathcal{C}$ and therefore is equal to $\mathcal{C}$. Hence ψ is surjective. It is also obvious that ψ is injective and therefore ψ is the required $*$-isomorphism. $\square$

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